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ON A CASE OF EXPLICIT SOLUTION OF A NONSTATIONARY INVERSE PROBLEM

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We examine the following inverse problem. Let the function $u(x, z, t)$ satisfy an equation of form

$$u_{tt} = a(z) \frac{\partial}{\partial z}(a(z) u_z) + P\left(z, i\frac{\partial}{\partial x}\right) u \tag{1}$$

in the halfspace $z > 0$, $x \in R^n$, $t \in (-\infty, \infty)$, where $P(z, k)$ is a polynomial of arbitrary degree in $k \in R^n$ with coefficients that are functions of z, and $a(z) > 0$. Let u be a solution of Eq. (1), satisfying the conditions

$$u|_{t<0} = 0, \quad au_z|_{z=0} = \delta(x, t) \tag{2}$$

whose Fourier transform with respect to x, $v(k, z, t)$, is differentiable with respect to k as many times as the degree of polynomial P.

N o t e 1. The general conditions ensuring the existence of such a solution are not known; however, it exists automatically if the differential operator in the right hand side of Eq. (1) is of second order and elliptic (then u is the solution of the mixed boundary-value problem for a second-order hyperbolic equation [1]).

We pose the problem: determine the coefficients of operator $P(z, i\partial / \partial x)$ if the function

$$f(x, t) = u(x, 0, t) \tag{3}$$

is known.

It turns out that all the coefficients of polynomial $P(z, k)$ can be uniquely determined from the function $f(x, t)$ if the coefficient $a(z)$ is known. If a is not known, the coefficients of polynomial P are determined uniquely as functions of some coordinate y, i.e., of an unknown monotonic function of z. It is not possible to find the interrelated functions $a(z)$ and $y(z)$ (we note, however, that if $a(z)$ and the coefficients of polynomial P are not independent, then the function a can be found from f; see Note 3). In order to determine the coefficients of polynomial P it is sufficient to know in both cases only the first coefficients $g_m(t)$ of the expansion of the Fourier transform $g(k, t)$ of function $f(x, t)$ in powers of k, viz., the variable dual to x; we need to know as many of these coefficients as the degree of polynomial P (cf. [2 — 4]). The assertion made here is easily proved by the method suggested in an article by the author (*). It turns out that the following statement is true: if it is known that polynomial P has a free term equal to zero, i.e., $P(z, 0) = 0$, then all the succeeding coefficients of P are expressed in terms of functions $g_m(t)$ by means of simple explicit formulas.

*) Blagoveshchenskii, A.S., One-dimensional inverse boundary-value problem for a second-order hyperbolic equation. Zap. Nauchn. Seminarov Leningrad. Otdel. Mat. Inst., Vol.15, 1969.

Let us derive these formulas in the case when x is a one-dimensional variable and the degree of polynomial P is two. The extension of the derivation to more complicated situations does not tell us anything essentially new. From the formulas presented below it follows that the coefficients required depend on the second derivatives of the functions; $g_m(t)$ is stable in the metric of C. Thus, instead of z we introduce the new coordinate

$$y = \int_0^z (a(z))^{-1}\, dz$$

Then, passing to the Fourier transform, we obtain the problem

$$\begin{gathered} v_{tt} = v_{yy} + (bk + ck^2)\, v \\ v|_{t<0} = 0, \quad v_y|_{y=0} = \delta(t), \quad v|_{y=0} = g(k, t) \end{gathered} \tag{4}$$

Replacing function $v(k, y, t)$ by its expansion $v = v_0 + kv_1 + k^2v_2 + o(k^2)$ in powers of k, we obtain a chain of problems. For $v_0(y, t)$ we have

$$v_{0tt} = v_{0yy}, \quad v_0|_{t<0} = 0, \quad v_{0y}|_{y=0} = \delta(t)$$

Hence $v_0 = -\varepsilon(t - y)$ and $g_0(t) = -\varepsilon(t)$ (the necessary condition for the solvability of the inverse problem). Here $\varepsilon(t)$ is the Heaviside function.

For $v_1(y, t)$ we have

$$v_{1tt} = v_{1yy} + bv_0, \quad v_1|_{t<0} = 0, \quad v_{1y}|_{y=0} = 0$$

Hence for $t > y$

$$v_1(y, t) = -\frac{1}{2}\int_0^{\frac{1}{2}(t-y)} b(\eta)(t - y - 2\eta)\, d\eta - \frac{t-y}{2}\int_0^y b(\eta)\, d\eta - \frac{1}{2}\int_y^{\frac{1}{2}(t+y)} b(\eta)(t + y - 2\eta)\, d\eta \overset{\text{def}}{=} Mb \tag{5}$$

$$v_1(0, t) = g_1(t) = -\int_0^{\frac{1}{2}t} b(\eta)(t - 2\eta)\, d\eta \tag{6}$$

From formula (6) we see that $g_1 = g_1^{\cdot} = 0$ at $t = 0$. Differentiating (6) twice with respect to t, we find

$$-2g_1^{\cdot\cdot}(t) = b(t/2) \tag{7}$$

Substituting expression (7) into (5), we find the representation for v_1

$$v_1 = \tfrac{1}{2}[g_1(t + y) + g_1(t - y) - g_1(2y)]\varepsilon(t - y)$$

For $v_2(y, t)$ we have

$$v_{2tt} = v_{2yy} + cv_0 + bv_1, \quad v_2|_{t<0} = 0, \quad v_{2y}|_{y=0} = 0$$

Analogously to the preceding we obtain

$$v_2(y, t) = Mc + Nbv_1 \tag{8}$$

where Nbv_1 can be explicitly expressed in terms of b and v_1, and, consequently, by virtue of (5) and (7), in terms of g_1. Setting $y = 0$ in (8) and differentiating twice with respect to t, after some manipulations we obtain

$$c\left(\frac{t}{2}\right) = -2g_2^{\cdot\cdot}(t) - g_1^{\cdot 2}(t) - \int_0^t d\tau\, g_1^{\cdot\cdot}(\tau)\, g_1^{\cdot}(t - \tau) \tag{9}$$

Formulas (7) and (9) yield the desired answer.

N o t e 2. The condition $P(z, 0) = 0$ can be relaxed; in its stead we can require that the coefficients of polynomial P be linearly-dependent functions of z and that $P(z, k_0) \equiv 0$ for some k_0. Then the coefficients of P are found explicitly from the coefficients of the expansion of $g(k, t,$ in powers of $k - k_0$.

N o t e 3. We consider the following example. As is well known [5], the acoustic equation has the form

$$u_{tt} = \rho a^2 \operatorname{div} (\rho^{-1} \operatorname{grad} u) \tag{10}$$

where ρ is the density of the medium and a is the velocity of sound. Let the medium be such that $\rho = \lambda/a$, a is a function of z, $\lambda = \text{const}$. Then in the two-dimensional case (as also in the three-dimensional case) the inverse problem leads to a problem of form (4) wherein $b \equiv 0$ and $c = -a^2$. From formula (9) it follows that $a^2(t/2) = 2g_2''(t)$ (g_1 of necessity equals zero)

$$z = \int_0^y a(t)\, dt$$

N o t e 4. Equation (10) describes as well (see [2]) the propagation of elastic waves of type SH in a layered-inhomogeneous medium if in it we replace ρ^{-1} by μ and ρa^2 by $(\rho')^{-1}$, where $\rho'(z)$ and $\mu(z)$ are, respectively, the density and the Lamé parameter of the elastic medium. The results obtained above are applicable if the medium is such that $\rho' \mu = \text{const}$.

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